Explicit solutions to certain inf max problems from Turán power sum theory

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Abstract

In a previous paper [1] we proved that

$$\sqrt{n} \leq \inf_{|z_k|\geq 1} \max_{\nu=1,\ldots,n^2} \left|\sum_{k=1}^{n} z_k^{\nu}\right| \leq \sqrt{n+1}$$

when $n+1$ is prime. In this paper we prove that

$$\inf_{|z_k|=1} \max_{\nu=1,\ldots,n^2-n} \left|\sum_{k=1}^{n} z_k^{\nu}\right| = \sqrt{n-1}$$

when $n-1$ is a prime power, and

$$\inf_{|z_k|\geq 1} \max_{\nu=1,\ldots,n^2-i} \left|\sum_{k=1}^{n} z_k^{\nu}\right| = \sqrt{n} \qquad (i=2,\ldots,n-1)$$

when $n \geq 3$ is a prime power. We give explicit constructions of n-tuples $(z_1,\ldots,z_n)$ which we prove are global minima for these problems. These are two of the few times in Turán power sum theory where solutions in the inf max problem can be explicitly calculated.

1 Introduction

In his paper [15] Turán shows that

$$\inf_{|z_k|\geq 1} \max_{\nu=1,\ldots,n} \left|\sum_{k=1}^{n} z_k^{\nu}\right| = 1.$$

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Furthermore he gives an explicit construction

$$z_k = e\left(\frac{k}{n+1}\right) \qquad\qquad \left(e(x) = e^{2\pi i x}\right)$$

which yields a global minimum for the problem. He also showed that this is essentially (up to rearrangements of the z_k and multiplication with a fixed unimodular constant) the only global minimum. Another result given by Cassels [7] is

$$\inf_{z_1=1,\, z_k\in\mathbb{C}} \max_{\nu=1,\dots,2n-1} \left|\sum_{k=1}^{n} z_k^\nu\right| = 1. \tag{1}$$

From this result it is clear that a global minimum can be given by $z_k = 0$, for $k = 2, \dots, n$. In this case there are a lot of different solutions though. It has been shown by Dancs [9] that if we replace $2n-1$ by $3n-3$ there only exist the trivial solution. If we instead replace $2n-1$ by $3n-4$ there are already infinitely many different solutions to this problem.

Another problem studied in Turán's book [16] is

$$\inf_{z_1=1,\, z_k\in\mathbb{C}} \max_{\nu=1,\dots,n} \left|\sum_{k=1}^{n} z_k^\nu\right|.$$

Atkinson [2] showed that this quantity lies in the interval $[1/6, 1]$, Biró ([4] and [3]) showed that it lies in the interval $[1/2+q, 5/6]$ for some $q > 0$ and a sufficiently large n. In this case there exist no simple solution to the inf max problem, and in fact Cheer-Goldston [8] used a computer to numerically obtain the minimal systems for small values of n. This is a typical situation in Turán power sum theory. For most problems we have to be satisfied with inequalities and have little hope of obtaining an equality.

In his book [16] Turán had a number of open problems. As problem 10 he proposed the study of the quantity

$$\inf_{|z_k|\geq 1} \max_{\nu=1,\dots,n^2} \left|\sum_{k=1}^{n} z_k^\nu\right|. \tag{2}$$

In a previous paper [1] we proved the strong inequality

$$\sqrt{n} \leq \inf_{|z_k|\geq 1} \max_{\nu=1,\dots,n^2} \left|\sum_{k=1}^{n} z_k^\nu\right| \leq \sqrt{n+1} \tag{3}$$

which is valid whenever $n+1$ is prime. The proof relied on an explicit construction due to Hugh Montgomery (see Turán [16] p. 83). It would be interesting to find a true global minimum for the problem, and we may ask:

Problem 1. Does Montgomery's construction (for some sufficiently large n such that $n+1$ is prime) give a true global minimum for the quantity (2), or equivalently can the upper inequality be replaced by an equality in eq. (3)?

We will not be able to answer this question in this paper. We will however be able to answer the corresponding question in some closely related problems. We will use explicit constructions to prove the following theorems:

Theorem 1. *Suppose that $n-1$ is a prime power. Then*

$$\inf_{|z_k|=1} \max_{\nu=1,\dots,n^2-n} \left| \sum_{k=1}^{n} z_k^{\nu} \right| = \sqrt{n-1}.$$

Furthermore an explicit n-tuple $(z_1,\dots,z_n)$ which gives a global minimum is given by Theorem 3 and eq. (4)*.*

Theorem 2. *Let $n \geq 3$ be a prime power, and let $2 \leq i \leq n-1$. Then*

$$\inf_{|z_k|\geq 1} \max_{\nu=1,\dots,n^2-i} \left| \sum_{k=1}^{n} z_k^{\nu} \right| = \sqrt{n}.$$

Furthermore the explicit n-tuple $(z_1,\dots,z_n)$ given by Theorem 4 and eq. (7) *provides a global minimum for the problem.*

2 Fabykowski's construction

In our paper [1] we also used a construction of Fabrykowski [11] to prove some results closely related to eq. (3). The construction by Fabrykowski depends of the existence of a perfect difference set

Theorem 3. (Singer) *Let $n-1$ be a prime power. Then there exists integers $a_1,\dots,a_n$ such that the integers $a_i - a_j$ for $i \neq j$ form all non zero residues* $\bmod\ n^2-n+1$.

of Singer [13]. By choosing

$$z_k = e\left(\frac{a_k}{n^2-n+1}\right), \qquad \left(e(x) = e^{2\pi i x}\right) \tag{4}$$

we see that

$$\begin{aligned}\left|\sum_{k=1}^{n} z_k^{\nu}\right|^2 &= n + \sum_{i\neq j} e\left(\frac{\nu(a_j-a_i)}{n^2-n+1}\right)\\ &= n-1+\sum_{j=1}^{n^2-n+1} e\left(\frac{\nu j}{n^2-n+1}\right)\\ &= \begin{cases} n-1, & n^2-n+1 \not|\nu,\\ n^2, & n^2-n+1|\nu.\end{cases}\end{aligned}$$

Hence we obtain (Andersson [1] Lemma 3)

Lemma 1. *Let n be a prime power. There exists an n-tuple of unimodular complex numbers such that*

$$\left|\sum_{k=1}^{n} z_k^{\nu}\right| = \begin{cases} \sqrt{n-1}, & n^2-n+1 \not|\nu,\\ n, & n^2-n+1|\nu.\end{cases}$$

In Andersson [1] we used a lemma of Cassels to prove that ([1], Corollary 1)

$$\inf_{|z_k|\geq 1}\max_{\nu=1,\ldots,2nm-m(m+1)+1}\left|\sum_{k=1}^{n} z_k^{\nu}\right| \geq \sqrt{m}. \qquad (1\leq m\leq n) \tag{5}$$

Together with Lemma 1 this implies ([1], Proposition 1 (*ii*))

$$\sqrt{n-2} \leq \inf_{|z_k|\geq 1}\max_{\nu=1,\ldots,n^2-n}\left|\sum_{k=1}^{n} z_k^{\nu}\right| \leq \sqrt{n-1}. \tag{6}$$

There exist another way of getting lower bounds for the inf max problem, which is a theorem independently proved by Newman, Cassels and Szalay [14] (see Theorem 7.3 in Turán [16]). The result is more general than eq. (5) since it does not assume that we consider the pure power sum problem (We can also have coefficients $b_j > 0$). However it is also less general since it assumes that $|z_k| = 1$. We will state the result for the pure power sum case:

Lemma 2. *Suppose that z_k are unimodular complex numbers, and $c \geq 1$ is an integer. Then*

$$\max_{1\leq\nu\leq cn}\left|\sum_{k=1}^{n} z_k^{\nu}\right| \geq \sqrt{\frac{cn-n+1}{c}}.$$

In the special case $c = n-1$ we obtain the lower bound

$$\max_{1\le\nu\le n^2-n}\left|\sum_{k=1}^{n} z_k^{\nu}\right| \ge \sqrt{n-1}$$

and by combining this with Lemma 1 we obtain a proof of Theorem 1. Natural problems to ask are:

Problem 2. Does all global minima of the min max problem in Theorem 1 arise from Fabrykowski's construction (and multiplication of a fixed unimodular constant)?

Problem 3. Can the condition $|z_k| = 1$ in Theorem 1 be replaced by $|z_k| \ge 1$?

By the result of Blanksby [5] (or it can be proved immediately from Turán's second main theorem [16] Chapter 8) we can give a partial answer to Problem 3:

Proposition. *There exist a constant $C > 0$ such that for every $n \ge 1$ we have that if $(z_1, \ldots, z_n)$ is a global minimum for the problem in eq.* (6)*, then $|z_k| \le (1 + C/n)$.*

3 A new construction and a theorem of Bose

This theorem of Singer is also used to construct Golomb rulers and Sidon sets (For a survey, see Dimitromanolakis [10]), and in fact Montgomery's construction is in this setting equivalent to Ruzsa's construction (see [12] Theorem 2) which is also used to construct Golomb rulers and Sidon sets. There also exists a third construction of Bose [6] which is used to construct Golomb rulers that has hitherto not been used in the corresponding power sum problem. We will now see what this construction will yield when applied to the power sum problem.

We first state a result taken from Bose [6]:

Theorem 4. (Bose) *Let n be a prime power. There exists integers $b_1, \ldots, b_n$ such that the residues $b_i - b_j$ for $i \ne j$ form all residues* $\mod n^2-1$ *which are not divisible by $n+1$.*

By choosing

$$z_k = e\left(\frac{b_k}{n^2-1}\right), \qquad \left(e(x) = e^{2\pi i x}\right) \tag{7}$$

we see that

$$\begin{aligned}
\left|\sum_{k=1}^{n} z_k^{\nu}\right|^2 &= n + \sum_{i\neq j} e\left(\frac{\nu(b_j-b_i)}{n^2-1}\right) \\
&= n + \sum_{j=1}^{n^2-1} e\left(\frac{\nu j}{n^2-1}\right) - \sum_{j=1}^{n-1} e\left(\frac{\nu j}{n-1}\right) \\
&= \begin{cases} n, & (n-1)\not|\nu, \\ 1, & (n-1)|\nu. \end{cases} \qquad (\nu = 1, \ldots, n^2-2)
\end{aligned}$$

Hence we obtain

Lemma 3. *Let n be a prime power. There exists an n-tuple of unimodular complex numbers such that for $\nu = 1, \ldots, n^2-2$ one has that*

$$\left|\sum_{k=1}^{n} z_k^{\nu}\right| = \begin{cases} \sqrt{n}, & (n-1)\not|\nu, \\ 1, & (n-1)|\nu. \end{cases} \tag{8}$$

By combining this with the choice $m = n$ in eq. (5) (see also Andersson [1], Corollary 3) we obtain a proof of Theorem 2. As in Problem 2 we may ask

Problem 4. Does all global minima of the min max problem in Theorem 2 arise from Bose's construction (and multiplication of a fixed unimodular constant)?

All problems 1-4 would be interesting to investigate numerically as in Cheer-Goldston [8] for small values of n.

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